

# ON A TYPE OF EXPONENTIAL FUNCTIONAL EQUATION AND ITS SUPERSTABILITY IN THE SENSE OF GER

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ABSTRACT. In this paper, we deal with a type exponential functional equation as follows

$$f(xy) = f(x)^{g(y)},$$

where  $f$  and  $g$  are two real valued functions on a commutative semigroup. Our aim of this paper is to prove that the above functional equation in the sense of Ger is superstable.

## 1. INTRODUCTION

In 1979, a type of stability was observed by J. Baker, J. Lawrence and F. Zorzitto [5]. Indeed, they proved that if a function is approximately exponential, then it is either a true exponential function or bounded. Then the exponential functional equation is said to be superstable. This result was the first result concerning the superstability phenomenon of functional equations. Later, J. Baker [4] (see also [1, 3, 6, 13]) generalized this famous result as follows:

Let  $(S, \cdot)$  be an arbitrary semigroup, and let  $f$  map  $S$  into the field  $C$  of all complex numbers. Assume that  $f$  is an approximately exponential function, i.e., there exists a nonnegative number  $\varepsilon$  such that

$$\|f(x \cdot y) - f(x)f(y)\| \leq \varepsilon$$

for all  $x, y \in S$ . Then  $f$  is either bounded or exponential.

The result of Baker, Lawrence and Zorzitto [5] was generalized by L. Székelyhidi [12] in another way and he obtained the following result.

**Theorem 1.1.** *Let  $(G, \cdot)$  be an Abelian group with identity 1 and let  $f, m : G \rightarrow \mathbb{C}$  be functions such that there exist functions  $M_1, M_2 : \rightarrow [0, \infty)$  with*

$$\|f(x \cdot y) - f(x)m(y)\| \leq \min\{M_1(x), M_2(y)\}$$

*for all  $x, y \in G$ . Then either  $f$  is bounded or  $m(x \cdot y) = m(x)m(y)$  and  $f(x) = f(1)g(x)$  for all  $x \in G$ .*

Also, M. Alimohammady and A. Sadeghi [2] proved a superstability result for the Cauchy equation. Moreover, they gave a partial affirmative answer to problem 18, in the thirty-first ISFE.

R. Ger pointed out that the superstability phenomenon of the exponential equation is caused by the fact that the natural group structure in the range space is

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disregarded, and he suggested a new type of stability for the exponential equation (ref. ([9]):

$$(1.1) \quad \left| \frac{f(x+y)}{f(x)+f(y)} - 1 \right| \leq \delta$$

If, for each function  $f : G \rightarrow E \setminus \{0\}$  satisfying the inequality (1.1) for some  $\delta > 0$  and for all  $x, y \in G$ , where  $E$  is a real Banach space, there exists an exponential function  $M : G \rightarrow E \setminus \{0\}$  such that

$$(1.2) \quad \left| \frac{f(x)}{M(x)} - 1 \right| \leq \phi(\delta) \quad \text{and} \quad \left| \frac{M(x)}{f(x)} - 1 \right| \leq \psi(\delta)$$

for all  $x \in G$ , where  $\phi(\delta)$  and  $\psi(\delta)$  depend on  $\delta$  only, then the exponential functional equation is said to be stable in the sense of Ger.

Every complex-valued function of the form  $f(x) = a^x$  ( $x \in \mathbb{C}$ ), where  $a > 0$  is a given number, is a solution of the functional equation

$$(1.3) \quad f(xy) = f(x)^y$$

Hence, the above functional equation may be regarded as a variation of the exponential functional equation.[11]

S.-M. Jung [10] proved the stability of the equation (1.3) in the sense of Ger and he obtained the following result.

**Theorem 1.2.** *Let  $\delta \in (0, 1)$  be a given number. If a function  $f : (0, \infty) \rightarrow (0, \infty)$  satisfies the inequality*

$$(1.4) \quad \left| \frac{f(xy)}{f(x)^y} - 1 \right| \leq \delta$$

*for all  $x, y > 0$ , then there exists a unique constant  $a > 0$  such that*

$$(1 - \delta)^{\alpha(x)} \leq \frac{a^x}{f(x)} \leq (1 + \delta)^{\alpha(x)},$$

*in which  $\alpha(x) = \sum_{n=1}^{\infty} (\prod_{i=0}^{n-1} x^{2^i})^{-1}$  for all  $x > 1$ .*

In this paper, we deal with a type of exponential functional equation as follows

$$(1.5) \quad f(xy) = f(x)^{g(y)}$$

where  $f$  and  $g$  are two real valued functions on a commutative semigroup. We prove that the above functional equation in the sense of Ger is superstable.

For the readers convenience and explicit later use, we will recall a fundamental results in fixed point theory.

**Definition 1.3.** The pair  $(X, d)$  is called a generalized complete metric space if  $X$  is a nonempty set and  $d : X^2 \rightarrow [0, \infty]$  satisfies the following conditions:

- (1)  $d(x, y) \geq 0$  and the equality holds if and only if  $x = y$ ;
- (2)  $d(x, y) = d(y, x)$ ;
- (3)  $d(x, z) \leq d(x, y) + d(y, z)$ ;
- (4) every d-Cauchy sequence in  $X$  is d-convergent.

for all  $x, y \in X$ .

Note that the distance between two points in a generalized metric space is permitted to be infinity.

**Theorem 1.4.** [8] *Let  $(X, d)$  be a generalized complete metric space and  $J : X \rightarrow X$  be strictly contractive mapping with the Lipschitz constant  $L$ . Then for each given element  $x \in X$ , either*

$$d(J^n(x), J^{n+1}(x)) = \infty$$

*for all nonnegative integers  $n$  or there exists a positive integer  $n_0$  such that*

- (1)  $d(J^n(x), J^{n+1}(x)) < \infty$ , for all  $n \geq n_0$ ;
- (2) the sequence  $\{J^n(x)\}$  converges to a fixed point  $y^*$  of  $J$ ;
- (3)  $y^*$  is the unique fixed point of  $J$  in the set  $Y = \{y \in X : d(J^{n_0}(x), y) < \infty\}$ ;
- (4)  $d(y, y^*) \leq \frac{1}{1-L}d(J(y), y)$ .

## 2. MAIN RESULTS

Throughout this Section, assume that  $S$  is an arbitrary commutative semigroup with identity 1 and  $\psi : S^2 \rightarrow [0, \infty)$  is a function. Let  $g : S \rightarrow \mathbb{R}$  be a function, then we define the set  $N_g$  with

$$N_g = \{a \in S : |g(a)| > 1\}.$$

**Theorem 2.1.** *Suppose that  $f : S \rightarrow (0, \infty)$  and  $g : S \rightarrow \mathbb{R}$  are two functions and satisfies the inequality*

$$(2.1) \quad 0 \leq \frac{f(xy)}{f(y)^{g(x)}} - 1 \leq \psi(x, y)$$

*for all  $x, y \in S$ . If  $N_g \neq \emptyset$  and  $\psi(x, ay) \leq \psi(x, y)$  for all  $x, y \in S$  and  $a \in N_g$ , then either  $g$  is bounded or*

$$(2.2) \quad f(xy) = f(y)^{g(x)}$$

*for all  $x, y \in S$ .*

**Proof.** From 2.1, we have

$$(2.3) \quad 1 \leq \frac{f(xy)}{f(y)^{g(x)}} \leq 1 + \psi(x, y)$$

for all  $x, y \in S$ . Then,

$$(2.4) \quad \left| \ln \frac{f(xy)}{f(y)^{g(x)}} \right| \leq \ln(1 + \psi(x, y))$$

or

$$(2.5) \quad |\ln f(xy) - g(x) \ln(f(y))| \leq \ln(1 + \psi(x, y)) \leq 1 + \psi(x, y)$$

for all  $x, y \in S$ . Set  $\tilde{\psi}(x, y) := 1 + \psi(x, y)$  for all  $x, y \in S$ , then its obvious that

$$(2.6) \quad \tilde{\psi}(x, ya) \leq \tilde{\psi}(x, y)$$

for all  $x, y \in S$  and  $a \in N_g$ . Let  $a \in N_g$  be fixed and from (2.5), we get

$$(2.7) \quad |\ln f(ay) - g(a) \ln(f(y))| \leq \tilde{\psi}(a, y)$$

for all  $y \in S$ . Let us consider the set  $A := \{g : S \rightarrow (0, \infty)\}$  and introduce the generalized metric on  $A$ :

$$d(g, h) = \sup_{y \in S} \frac{\|g(y) - h(y)\|}{\tilde{\psi}(a, y)}.$$

It is easy to show that  $(A, d)$  is complete metric space. Now we define the function  $J_a : A \rightarrow A$  with

$$J_a(h(y)) = \frac{1}{g(a)}h(ay)$$

for all  $h \in A$  and  $y \in S$ . So

$$\begin{aligned} d(J_a(u), J_a(h)) &= \sup_{y \in S} \frac{\|u(ay) - h(ay)\|}{|g(a)|\tilde{\psi}(a, y)} \\ &\leq \sup_{y \in S} \frac{\|u(ay) - h(ay)\|}{|g(a)|\tilde{\psi}(a, ay)} = \frac{1}{|g(a)|}d(u, h) \end{aligned}$$

for all  $u, h \in A$ , that is  $J$  is a strictly contractive selfmapping of  $A$ , with the Lipschitz constant  $L = \frac{1}{|g(a)|}$ . From (2.7), we get

$$\left\| \frac{\ln f(ay)}{g(a)} - \ln f(y) \right\| \leq \frac{\tilde{\psi}(a, y)}{|g(a)|}$$

for all  $y \in S$ , which says that  $d(J(\ln f), \ln f) \leq L < \infty$ . By Theorem (1.4), there exists a mapping  $T_a : S \rightarrow (0, \infty)$  such that

(1)  $T_a$  is a fixed point of  $J$ , i.e.,

$$(2.8) \quad T_a(ay) = g(a)T_a$$

for all  $y \in S$ . The mapping  $T_a$  is a unique fixed point of  $J$  in the set  $\tilde{A} = \{h \in A : d(\ln f, h) < \infty\}$ .

(2)  $d(J^n(\ln f), T_a) \rightarrow 0$  as  $n \rightarrow \infty$ . This implies that

$$T_a(y) = \lim_{n \rightarrow \infty} \frac{\ln f(a^n y)}{g(a)^n}$$

for all  $x \in S$ .

(3)  $d(\ln f, T_a) \leq \frac{1}{1-L}d(J(\ln f), \ln f)$ , which implies,

$$d(\ln f, T_a) \leq \frac{1}{|g(a)| - 1}.$$

From (2.7), its easy to show that following inequality

$$(2.9) \quad \|\ln f(a^n y) - g(a)^n \ln f(y)\| \leq \sum_{i=0}^{n-1} \tilde{\psi}(a, ya^i) |g(a)|^{n-1-i}$$

for all  $y \in S$  and  $n \in \mathbb{N}$ . Now since  $\tilde{\psi}(a, ya) \leq \tilde{\psi}(a, y)$  for all  $y \in S$ , so

$$\tilde{\psi}(a, ya^m) \leq \tilde{\psi}(a, y)$$

for all  $x \in S$  and  $m \in \mathbb{N}$ , thus from (2.9), we obtain

$$(2.10) \quad \|\ln f(ya^n) - g(a)^n \ln f(y)\| \leq \tilde{\psi}(a, y) \frac{|g(a)|^n - 1}{|g(a)| - 1}$$

for all  $y \in S$ . With this inequality (2.10), we prove that  $T_a = T_b$  for each  $a, b \in N_g$ . We have from inequality (2.10)

$$(2.11) \quad \|\ln f(ya^n) - g(a)^n \ln f(y)\| \leq \tilde{\psi}(a, y) \frac{|g(a)|^n - 1}{|g(a)| - 1}$$

$$(2.12) \quad \|\ln f(yb^n) - g(b)^n \ln f(y)\| \leq \tilde{\psi}(b, y) \frac{|g(b)|^n - 1}{|g(b)| - 1}$$

for all  $y \in S$ . On the replacing  $y$  by  $yb^n$  in (2.11) and  $y$  by  $ya^n$  in (2.12)

$$\begin{aligned} \|\ln f(y(ab)^n) - g(a)^n \ln f(yb^n)\| &\leq \tilde{\psi}(a, y) \frac{|g(a)|^n - 1}{|g(a)| - 1} \\ \|\ln f(y(ab)^n) - g(b)^n \ln f(ya^n)\| &\leq \tilde{\psi}(b, y) \frac{|g(b)|^n - 1}{|g(b)| - 1}. \end{aligned}$$

Thus,

$$\|g(a)^n \ln f(yb^n) - g(b)^n \ln f(ya^n)\| \leq \tilde{\psi}(a, y) \frac{|g(a)|^n - 1}{|g(a)| - 1} + \tilde{\psi}(b, y) \frac{|g(b)|^n - 1}{|g(b)| - 1}$$

and dividing by  $|g(a)^n g(b)^n|$

$$\begin{aligned} \left\| \frac{\ln f(ya^n)}{g(a)^n} - \frac{\ln f(yb^n)}{g(b)^n} \right\| &\leq \\ \frac{\tilde{\psi}(a, y)}{(|g(b)| - 1)|g(a)|^n} \left(1 - \frac{1}{|g(b)|^n}\right) &+ \frac{\tilde{\psi}(b, y)}{(|g(a)| - 1)|g(b)|^n} \left(1 - \frac{1}{|g(b)|^n}\right) \end{aligned}$$

and letting  $n$  to infinity, we obtain  $T_a(y) = T_b(y)$  for all  $y \in S$ . Therefore, there a unique function  $T$  such that  $T = T_a$  for every  $a \in N_g$  and

$$\|\ln f(y) - T(y)\| \leq \frac{\tilde{\psi}(a, y)}{|g(a)| - 1}$$

for all  $y \in S$  and  $a \in N_g$ . Since  $a \in N_g$  is a arbitrary element, so

$$\|\ln f(y) - T(y)\| \leq \inf_{a \in N_g} \frac{\tilde{\psi}(a, y)}{|g(a)| - 1}$$

for all  $y \in S$ . Now if  $g$  be a unbounded function, we get  $T = \ln f$ .

Let  $x, y \in S$  and  $a \in N_g$  be three arbitrary fixed elements, from (2.7)

$$\|\ln f(xya^n) - g(x) \ln f(ya^n)\| \leq \tilde{\psi}(x, ya^n)$$

and dividing by  $|g(a)|^n$ ,

$$\left\| \frac{\ln f(xya^n)}{g(a)^n} - g(x) \frac{\ln f(ya^n)}{g(a)^n} \right\| \leq \frac{\tilde{\psi}(x, ya^n)}{|g(a)|^n} \leq \frac{\tilde{\psi}(x, y)}{|g(a)|^n}$$

and letting  $n$  to infinity, we get

$$(2.13) \quad f(xy) = f(y)^{g(x)}$$

for all  $x, y \in S$ . The proof is complete.

With the above Theorem, its easy to obtain the following results.

**Corollary 2.2.** *Let  $\delta > 0$  be a given number. Assume that  $f : S \rightarrow (0, \infty)$  and  $g : S \rightarrow \mathbb{R}$  satisfies the inequality*

$$(2.14) \quad 0 \leq \frac{f(xy)}{f(y)^{g(x)}} - 1 \leq \delta$$

for all  $x, y \in S$ , then either  $g$  is bounded or

$$(2.15) \quad f(xy) = f(y)^{g(x)}$$

for all  $x, y \in S$ .

**Corollary 2.3.** *Let  $\delta > 0$  be a given number. If a function  $f : (0, \infty) \rightarrow (0, \infty)$  satisfies the inequality*

$$(2.16) \quad 0 \leq \frac{f(xy)}{f(y)^x} - 1 \leq \delta$$

*for all  $x, y > 0$ , then*

$$(2.17) \quad f(xy) = f(y)^x$$

*for all  $x, y > 0$ .*

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